B.Sc. (Maths) part II
paper IV

Topic Differential eqn. of central orbit in polar and pedal form

Theorem A particle moves in an ellipse under force which is always directed towards its focus. Find the law of force and the velocity at any point of its path.

Answer:— The equation to the ellipse with focus as pole is

$$\frac{l}{r} = 1 + e\cos\theta$$

$$\Rightarrow \frac{1}{r} = \frac{1}{l} + \frac{e}{l}\cos\theta$$

$$\Rightarrow u = \frac{1}{l} + \frac{e}{l}\cos\theta \quad \left[\because \frac{1}{r} = u\right]$$

$$\therefore \frac{du}{d\theta} = -\frac{e}{l}\sin\theta$$

and $$\frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta$$

~~Hence~~ we have from diff. eqn. of path

$$P = h^2u^2\left[u + \frac{d^2u}{d\theta^2}\right]$$

$$= h^2u^2\left[\frac{1}{l} + \frac{e}{l}\cos\theta - \frac{e}{l}\cos\theta\right]$$

$$= \frac{h^2}{l}u^2 = \frac{\mu}{r^2} \quad \text{where } \mu = \frac{h^2}{l}$$

i.e. $h = \sqrt{\mu l}$

$$\therefore P \propto \frac{1}{r^2}$$

Thus the central force varies inversely as the square of the distance from the focus

Also, $v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$

$$= h^2\left[\left(\frac{1}{l} + \frac{e}{l}\cos\theta\right)^2 + \left(-\frac{e}{l}\sin\theta\right)^2\right]$$

$$= \mu l\left[\frac{1}{l^2} + \frac{e^2}{l^2}(\cos^2\theta + \sin^2\theta) + \frac{2e}{l^2}\cos\theta\right]$$

$$= \mu\left[\frac{2}{l} - \frac{1}{l} + \frac{e^2}{l} + \frac{2e}{l}\cos\theta\right]$$

$$= \mu\left[\frac{2}{l}(1+\cos\theta) - \frac{1}{l}(1-e^2)\right]$$

$$= \mu\left[\frac{2}{r} - \frac{1}{a}\right] \text{ as } l = \frac{b^2}{a} = a(1-e^2)$$

If T be the time taken by the particle to describe the whole arc of the ellipse then

$$\frac{1}{2}h \times T = \text{area of the ellipse} = \pi ab$$

i.e $T = \dfrac{2\pi ab}{h} = \dfrac{2\pi ab}{\sqrt{\mu l}} = \dfrac{2\pi ab}{\sqrt{\mu \frac{b^2}{a}}}$

$$= \frac{2\pi}{\sqrt{\mu}} a^{3/2}$$

$$\therefore T^2 = \frac{4\pi^2}{\mu} a^3$$

$$T \propto a^3$$

Hence the square of the periodic time of a particle varies as the square of the semi-major axis of the ellipse

Problem ① show that the force towards the pole under which describes the curve $r = \frac{l}{1-e\cos\theta}$ varies as $\frac{1}{r^2}$

Soln:- we have $r = \frac{l}{1-e\cos\theta}$

i.e. $\frac{1}{u} = \frac{l}{1-e\cos\theta}$ i.e. $u = \frac{1}{l}(1-e\cos\theta)$

$\therefore \frac{du}{d\theta} = \frac{e}{l}\sin\theta$ and $\frac{d^2u}{d\theta^2} = \frac{e}{l}\cos\theta$

we have $P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right)$

$= h^2u^2\left[\frac{1}{l}(1-e\cos\theta) + \frac{e}{l}\cos\theta\right]$

$= \frac{h^2}{l}\frac{1}{r^2}$

Hence $P \propto \frac{1}{r^2}$

Scanned with CamScanner